Advanced Quantum Mechanics 2
lecture 8
Charged particle in an electromagnetic field

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The Lorentz force acting on a charged particle $q$ moving with velocity $\vec{v}$ in an electric field $\vec{E}$ and magnetic field $\vec{B}$ is given by

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

Expressed in terms of the vector and scalar potentials, the electric field and magnetic field are:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial}{\partial t}\vec{A}, \quad \vec{B} = \vec{\nabla} \times \vec{A}$$

The electric and magnetic fields (physical observables) are gauge invariant, meaning they are invariant under the gauge transformations of the vector and scalar potentials:

$$\vec{A} \rightarrow \vec{A}' = \vec{A} + \vec{\nabla}\chi, \quad \phi \rightarrow \phi' = \phi - \frac{d\chi}{dt}$$ (1)

which means that the choice of the vector and scalar potential is not unique.
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Note that the above choice of electric and magnetic fields reduces the number of free components from 6 to just 4. This is obvious from the two equations of Maxwell: Gauss’s law for magnetic field $\mathbf{\nabla}.\mathbf{B} = 0$ and Faraday’s law: $\mathbf{\nabla} \times \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{B}$, where these equations give constraints on the components of $\mathbf{E}$ and $\mathbf{B}$, leaving only 4 independent components. For instance, choosing the magnetic field along the $z$ axis and the electric field to be zero, $\mathbf{E} = 0$ and $\mathbf{B} = (0, 0, B_z)$ we can fix $\phi = 0$. We then have several possible choices of $\mathbf{A}$: $\mathbf{A} = (B_z y, 0, 0)$, $\mathbf{A} = (0, -xB_z, 0)$ or $\mathbf{A} = \frac{1}{2}(B_z y, -B_z x, 0)$. Because of gauge invariance we can choose to impose further constraints on the potentials, i.e. fix a certain gauge. Most usefully is the Coulomb gauge in which we chose:

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\[ \vec{\nabla} \cdot \vec{A} = 0 \]
This of course reduces the number of degrees of freedom by one further unit, to just 3. In this gauge we have from the relation (1):

\[ \nabla \cdot \vec{A}' = \nabla \cdot \vec{A} + \nabla^2 \chi = 0 \]

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Pauli Hamiltonian and gauge invariance

We can then construct a classical Hamiltonian to account for the interaction of a charged particle with an electromagnetic field: \(^1\)

\[
H = \frac{1}{2m} \left( \vec{p} - q\vec{A} \right)^2 + q\phi
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Quantising this into a quantum mechanical operator yields the Pauli Hamiltonian:

\[
\hat{H} = \frac{1}{2m} \left( \hat{\vec{p}} - q\hat{\vec{A}} \right)^2 + q\phi
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In the case that the particle has a spin then we add a term to this Hamiltonian accounting for the interaction of the spin magnetic moment with the magnetic field:

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\hat{H} = \frac{1}{2m} \left( \hat{\vec{p}} - q\hat{\vec{A}} \right)^2 + q\phi - \frac{q}{2m} g \hat{\vec{S}} \cdot \vec{B}
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\(^1\)The construction of this hamiltonian requires first construction of the Lagrangian which results in the Lorentz force equation. From this equation it is then straightforward to obtain the Hamiltonian.
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with \( g \) the **gyromagnetic ratio** of the interacting particle. In fact the magnetic moment corresponding to the spin operator is:

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\hat{\mu} = \frac{q}{2m} g \hat{S}
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and the Hamiltonian (potential energy) for the interaction of the magnetic field with the magnetic moment of the spin is \( -\hat{\mu}.\vec{B} \)

The above Hamiltonian is in fact the non-relativistic version of the Dirac equation. The interaction is merely obtained by the “minimal substitution” \( \hat{p} \rightarrow \hat{p}' = \hat{p} - q\vec{A} \) in addition to the potential energy term \( q\vec{A} \) (for electric field). This is analogous to changing ordinary derivative \( \hat{p} = -i\hbar \nabla \) with a “covariant derivative” which leaves the Hamiltonian invariant under the simultaneous gauge transformations of the vector.
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and scalar potential (eq. (1)) and local phase transformations of the wavefunctions:

$$\psi(\vec{r}, t) \rightarrow \psi'(\vec{r}, t) = e^{iq\chi(\vec{r}, t)/\hbar} \psi(\vec{r}, t)$$

Indeed under the simultaneous transformations the potential energy $q\phi$ and spin-magnetic field terms are not affected, and the remaining kinetic term transforms like:

$$\frac{1}{2m} \left( \vec{p} - q\vec{A} \right)^2 \psi(\vec{r}, t) \rightarrow \frac{1}{2m} \left( \vec{p} - q\vec{A}' \right)^2 \psi'(\vec{r}, t)$$

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\[
\left(-i\hbar \vec{\nabla} - q\vec{A} - q\vec{\nabla}\chi\right)^2 = \\
= -\hbar^2 \vec{\nabla}^2 + q^2 \vec{A}^2 + q^2 (\vec{\nabla}\chi)^2 + i\hbar q \vec{\nabla} \cdot \vec{A} + \\
+ 2i\hbar q \vec{A} \cdot \vec{\nabla} + i\hbar q (\vec{\nabla}^2 \chi) + 2i\hbar q (\vec{\nabla}\chi) \cdot \vec{\nabla} + 2q^2 \vec{A} \cdot (\vec{\nabla}\chi)
\]

(2)

Taking explicit derivatives we write:

\[
-\hbar^2 \vec{\nabla}^2 \left[e^{iq\chi(\vec{r},t)/\hbar} \psi(\vec{r},t)\right] = \\
e^{iq\chi(\vec{r},t)/\hbar} \left[-\hbar^2 \vec{\nabla}^2 + q^2 (\vec{\nabla}\chi)^2 - 2i\hbar q (\vec{\nabla}\chi) \cdot \vec{\nabla} - iq\hbar (\vec{\nabla}^2 \chi)\right] \psi(\vec{r},t)
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\left(-i\hbar \vec{\nabla} - q\vec{A} - q\vec{\nabla}\chi\right)^2 = \\
= -\hbar^2 \vec{\nabla}^2 + q^2 \vec{A}^2 + q^2 (\vec{\nabla}\chi)^2 + i\hbar q \vec{\nabla} \cdot \vec{A} + \\
+ 2i\hbar q \vec{A} \cdot \vec{\nabla} + i\hbar q (\vec{\nabla}^2 \chi) + 2i\hbar q (\vec{\nabla}\chi) \cdot \vec{\nabla} + 2q^2 \vec{A} \cdot (\vec{\nabla}\chi) \quad (2)
\]

Taking explicit derivatives we write:

\[
-\hbar^2 \vec{\nabla}^2 \left[ e^{iq\chi(\vec{r},t)/\hbar} \psi(\vec{r},t) \right] = \\
e^{iq\chi(\vec{r},t)/\hbar} \left[ -\hbar^2 \vec{\nabla}^2 + q^2 (\vec{\nabla}\chi)^2 - 2i\hbar (\vec{\nabla}\chi) \cdot \vec{\nabla} - iq\hbar (\vec{\nabla}^2\chi) \right] \psi(\vec{r},t)
\]
Pauli Hamiltonian and gauge invariance

and

\[ i q \hbar \vec{A} \cdot \vec{\nabla} \left[ e^{i q \chi(\vec{r}, t)/\hbar} \psi(\vec{r}, t) \right] = e^{i q \chi(\vec{r}, t)/\hbar} \left[ i q \hbar \vec{A} \cdot \vec{\nabla} - q^2 (\vec{A} \cdot \vec{\nabla} \chi) \right] \psi(\vec{r}, t) \]

\[ i q \hbar (\vec{\nabla} \chi) \cdot \vec{\nabla} \left[ e^{i q \chi(\vec{r}, t)/\hbar} \psi(\vec{r}, t) \right] = e^{i q \chi(\vec{r}, t)/\hbar} \left[ i q \hbar (\vec{\nabla} \chi) \cdot \vec{\nabla} - q^2 (\vec{\nabla} \chi)^2 \right] \psi(\vec{r}, t) \]

Substituting we find that:

\[ \frac{1}{2m} \left( \vec{p} - q \vec{A} \right)^2 \psi(\vec{r}, t) \rightarrow \frac{1}{2m} \left( \vec{p} - q \vec{A} \right)^2 \psi'(\vec{r}, t) = \frac{1}{2m} \left( \vec{p} - q \vec{A} \right)^2 \psi(\vec{r}, t) \]
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Pauli Hamiltonian and gauge invariance

Coupled with the transformation of $\phi \rightarrow \phi' = \phi - \frac{d}{dt} \chi$, and in the right-hand-side of the Schrödinger equation where we have:

$$i\hbar \frac{\partial}{\partial t} \left[ e^{i q \chi(\vec{r}, t)/\hbar} \psi(\vec{r}, t) \right] = e^{i q \chi(\vec{r}, t)/\hbar} \left[ i\hbar \frac{\partial}{\partial t} - q \frac{\partial \chi}{\partial t} \right] \psi(\vec{r}, t)$$

Then we see that the total Schrödinger remains nicely invariant under the simultaneous gauge transformation and local phase transformation of the wavefunction. This fact has profound implications in particle physics known as gauge theories.
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The Pauli-Schrödinger equation

The Schrödinger equation then reads:

\[ i\hbar \frac{\partial}{\partial t} |\psi\rangle = \hat{H} |\psi\rangle \]

where \( \hat{H} \) is the Pauli Hamiltonian given above. Since we are dealing with interaction including spin then the ket \( |\psi\rangle \) must belong to the space \( \mathcal{E}_s \otimes \mathcal{E}_\mathbf{r} \). Hence projected in position representation the wavefunction of the charged particle is in fact a spinor of the general form:

\[
|\psi\rangle = \begin{pmatrix}
|\psi_1\rangle \\
|\psi_2\rangle \\
\vdots \\
|\psi_n\rangle
\end{pmatrix}
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where \( n = 2s + 1 \) is the dimension of spin space.
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For an electron we have $n = 2$ so the wavefunction of the electron is a two-component spinor:

$$|\psi\rangle = \begin{pmatrix} |\psi_+\rangle \\ |\psi_-\rangle \end{pmatrix}$$

Projecting onto position representation we write:

$$\langle \vec{r} | \psi \rangle = \begin{pmatrix} \langle \vec{r} | \psi_+ \rangle \\ \langle \vec{r} | \psi_- \rangle \end{pmatrix}$$

and separating the spin variables from the spacial variables we write this as:

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where the vector $|\eta\rangle = \begin{pmatrix} \eta_+ \\ \eta_- \end{pmatrix}$ describes the spin state of the particle. For instance if $\eta_+ = 1$ and $\eta_2 = 0$ this means that our particle is in a state of definite $S_z$, which is in the $\uparrow$ state in our usual notation. Note that the vector $|\eta\rangle$ is normalised meaning that

$$|\eta_+|^2 + |\eta_-|^2 = 1$$

and that its components $(\eta_+, \eta_-)$ in spin state (of basis vectors $(|+\rangle, |-\rangle)$) are complex number that do not depend on the coordinates (and depend on time).
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For a **homogeneous** magnetic field \( \vec{B} = \text{const} \) (independent of time and space), substituting into the Schrödinger equation and using the fact that the spin vector is coordinate independent, in other words they commute with \( \hat{p}, \hat{A} \) and \( \phi \), we straightforwardly obtain their evolution in time keeping the only term with which they do not commute, i.e. the spin:

\[
i\hbar \frac{d}{dt} |\eta\rangle = -\frac{q}{2m} g \vec{B} \cdot \hat{S} |\eta\rangle
\]

This is the spin dynamics equation for a charged particle in a magnetic field. Writing this equation in matrix representation yields:

\[
\begin{bmatrix}
\frac{d}{dt} \begin{pmatrix} n_x^{\prime} \\ n_y^{\prime} \\ n_z^{\prime} \end{pmatrix} \\
\end{bmatrix} = -\mu_B \vec{B} \cdot \hat{\vec{S}} \begin{pmatrix} n_x^{\prime} \\ n_y^{\prime} \\ n_z^{\prime} \end{pmatrix}
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$$i\hbar \frac{d}{dt} \begin{pmatrix} \eta_+(t) \\ \eta_-(t) \end{pmatrix} = -\mu_B \vec{B} \cdot \hat{S} \begin{pmatrix} \eta_+(t) \\ \eta_-(t) \end{pmatrix}$$

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where \( \hat{\sigma} \) are the Pauli matrix and \( \mu_B = e\hbar/(2m_e c) \) is Bohr magneton for the electron (we specified to the case of an electron where \( q \rightarrow e \) and \( m \rightarrow m_e \) with a gyromagnetic ratio \( g_e = 2 \) for the electron). The above is the Pauli-Schrödinger equation for the spin dynamics of an electron in interaction with a magnetic field.

For the spin dynamics, for example choosing the magnetic field to be (uniform and) along the \( z \) direction we find:

\[
\frac{i\hbar}{dt} \left( \begin{array}{c} \eta_+(t) \\
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\]

where

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\sigma_z = \left( \begin{array}{cc}
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The Pauli-Schrödinger equation

Then it is straightforward to solve the resulting differential equations giving:

\[ \eta_+(t) = \eta_+(0)e^{i\omega t}, \quad \eta_-(t) = \eta_-(0)e^{-i\omega t}, \quad \omega = \frac{\mu_BB}{\hbar} \]

From this we can compute average values of spin components using usual quantum mechanics algebra for expectation values. The remaining part of the wavefunction (which is purely spatial) is written explicitly as:

\[ i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = \left[ \frac{1}{2m} \left( -i\hbar \vec{\nabla} - e\vec{A} \right)^2 + e\phi \right] \psi(\vec{r}, t) \]
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Aharonov-Bohm effect

Consider a charged particle in the vicinity of a long solenoid with steady flowing current. Now the magnetic field outside the solenoid is zero and inside the solenoid is uniform in magnitude and direction (and time). Now since there are no electric fields (the solenoid is uncharged) and since the current is steady (meaning $\partial \vec{A}/\partial t = 0$), then we find that $\vec{E} = -\vec{\nabla} \phi = 0$. Hence $\phi = \text{const.}$ Then since the magnetic field $\vec{B} = 0$ outside the solenoid then we must have $\vec{\nabla} \times \vec{A} = 0$, but inside the solenoid we have $\vec{B} =$ constant then inside the solenoid $\vec{\nabla} \times \vec{A} = \vec{B} = \text{const.}$ Furthermore using Stokes’s theorem for the line integral of the vector potential about any closed loop outside the solenoid which encloses the solenoid we obtain:

$$\oint \vec{A} \cdot d\vec{r} = \int \vec{\nabla} \times \vec{A} \cdot d\vec{S} = \int \vec{B} \cdot d\vec{S} = \Phi_m$$
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Aharonov-Bohm effect

where we have turned the line integral around the solenoid into a surface integral over a surface which encloses this loop (we can choose this to be just the disk enclosed by the circle). This surface however crosses the solenoid where the magnetic field is non-zero and the result of integration over this region is just the flux of the magnetic field over the solenoid. The vector potential outside the solenoid is non-zero, and assuming constant magnetic field the by symmetry we deduce that the vector potential is constant and rotational around the closed loop, which results in:

\[ \vec{A}(r) = \frac{\Phi}{2\pi r} \hat{u}_\phi \]

with \( \Phi = BA \) and \( A \) the surface area of the solenoid. We fixed the solenoid axis along the \( z \) axis.
Aharonov-Bohm effect

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Aharonov-Bohm effect

Let us find the solution the Schrödinger

\[ \imath \hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = \left[ \frac{1}{2m} \left( \imath \hbar \vec{\nabla} - e \vec{A} \right)^2 + e\phi \right] \psi(\vec{r}, t) \]

where the spin part is irrelevant since the magnetic field is zero in the region where electrons fly.

We will show that the solution to the Schrödinger equation for \( \vec{A} \neq 0 \) is related to that for \( \vec{A} = 0 \) by a phase factor:

\[ \psi(\vec{r}) = e^{ig(\vec{r})} \psi_0(\vec{r}) \]

where \( \psi_0(\vec{r}) \) is the solution to the Schrödinger equation in the case of zero vector potential (i.e. when the magnetic field is switched off), where

\[ g(\vec{r}) = \frac{q}{\hbar} \int_{\vec{r}_0}^{\vec{r}} \vec{A}(\vec{r}') \cdot d\vec{r}' \]
Aharonov-Bohm effect

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\[ i\hbar \frac{\partial \psi(\vec{r}, t)}{\partial t} = \left[ \frac{1}{2m} \left(-i\hbar \vec{\nabla} - e\vec{A} \right)^2 + e\phi \right] \psi(\vec{r}, t) \]

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Aharonov-Bohm effect

is a line integral from initial point \( \vec{r}_0 \) to the point of “measurement” of the wavefunction \( \vec{r} \). The initial point is arbitrary (which is due to arbitrary choice of gauge) and the result is path independent provided \( \vec{\nabla} \times \vec{A} = 0 \) along the path (i.e. the path must be outside the solenoid). By stokes's theorem the above line integral is the same for any two paths connecting the initial and final points provided that the loop resulting from the two paths does not enclose the solenoid (i.e. \( \vec{\nabla} \times \vec{A} \) must be zero for the surface enclosed between the two paths). If the surface encloses the solenoid then the result is path dependent in this case. To check this let us take the gradient of the proposed solution:

\[
\nabla \psi = (\nabla \psi_0) e^{ig(\vec{r})} + i(\nabla g(\vec{r})) \psi_0 e^{ig(\vec{r})}
\]

However we have:

\[
\nabla g(\vec{r}) = \frac{q}{\hbar} \vec{A}(\vec{r})
\]

\[
\nabla \psi(\vec{r}) = \frac{q}{\hbar} \nabla \int_{\vec{r_0}}^{\vec{r}} \vec{A}(\vec{r'}) \, d\vec{r'} - \frac{q}{\hbar} \vec{A}(\vec{r})
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Aharonov-Bohm effect

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$$\vec{\nabla} \psi = (\vec{\nabla} \psi_0) e^{ig(\vec{r})} + i(\vec{\nabla} g(\vec{r})) \psi_0 e^{ig(\vec{r})}$$

However we have:

$$\vec{\nabla} g(\vec{r}) = \frac{q}{\hbar} \oint \vec{A}(\vec{r}’) d\vec{r}’ = \frac{q}{\hbar} \vec{A}(\vec{r})$$

(1)
Aharonov-Bohm effect

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$$\vec{\nabla}g(\vec{r}) = q\hbar \int_{\vec{r}_0}^{\vec{r}} \vec{A}(\vec{r}') \cdot d\vec{r}' = q\hbar \vec{A}(\vec{r})$$
Aharonov-Bohm effect

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$$\nabla \psi = (\nabla \psi_0) e^{i\phi(r)} + i(\nabla \phi(r)) \psi_0 e^{i\phi(r)}$$

However, we have:

$$\nabla \psi_0 = \frac{2}{\hbar} \nabla \int_{\vec{r}'} A(\vec{r}') d\vec{r}' - \frac{2q}{\hbar} A(\vec{r})$$

To verify this, we need to calculate the integral $\int_{\vec{r}'} A(\vec{r}') d\vec{r}'$. This integral represents the flux of the magnetic field through the area enclosed by the path $\vec{r}$. If the solenoid does not enclose this area, the integral is zero, and the result is path independent. If the solenoid does enclose this area, the integral is non-zero, and the result is path dependent.
Aharonov-Bohm effect

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(http://theorique05.wordpress.com/f411) Advanced Quantum Mechanics 2 - lecture 8
Aharonov-Bohm effect

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Aharonov-Bohm effect

Thus substituting we find:

\[-i\hbar \vec{\nabla} \psi = (-i\hbar \vec{\nabla} \psi_0) e^{ig(\vec{r})} + q\vec{A}(\vec{r})\psi_0 e^{ig(\vec{r})}\]

\[\Rightarrow (\hat{p} - q\vec{A}(\vec{r})) \psi = (-i\hbar \vec{\nabla} \psi_0) e^{ig(\vec{r})}\]

Iterating:

\[\frac{1}{2m} (\hat{p} - q\vec{A}(\vec{r}))^2 \psi = -\frac{1}{2m} \hbar^2 (\vec{\nabla}^2 \psi_0) e^{ig(\vec{r})} = \frac{1}{2m} \left( \hat{p}^2 \psi_0 \right) e^{ig(\vec{r})}\]

however \(\psi_0\) is the solution in the zero vector potential case, meaning that it satisfies the Schrödinger equation:

\[i\hbar \frac{\partial \psi_0(\vec{r})}{\partial t} = \frac{1}{2m} \hat{p} \psi_0(\vec{r})\]

where we put \(\phi = 0\), but this equally works for \(\phi = \text{const.}\).
Aharonov-Bohm effect

Thus substituting we find:

\[-i\hbar \vec{\nabla} \psi = (-i\hbar \vec{\nabla} \psi_0)e^{ig(\vec{r})} + q\vec{A}(\vec{r})\psi_0e^{ig(\vec{r})}\]

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Iterating:

\[\frac{1}{2m}(\hat{\vec{p}} - q\vec{A}(\vec{r}))^2\psi = -\frac{1}{2m}\hbar^2(\vec{\nabla}^2 \psi_0)e^{ig(\vec{r})} = \frac{1}{2m}\left(\hat{\vec{p}}^2 \psi_0\right)e^{ig(\vec{r})}\]

however $\psi_0$ is the solution in the zero vector potential case, meaning that it satisfies the Schrödinger equation:

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Thus substituting we find:

\[-i\hbar \vec{\nabla} \psi = (-i\hbar \vec{\nabla} \psi_0) e^{ig(\vec{r})} + q\vec{A}(\vec{r})\psi_0 e^{ig(\vec{r})}\]

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Aharonov-Bohm effect

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Hence (and since the phase factor is time independent) we immediately see that:

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Aharonov-Bohm effect

Consider the double slit experiment with electrons where we place a long solenoid between the slits as shown in the figure.
Aharonov-Bohm effect

The indicated paths in the figure have different phase contributions to the wave function of the electron (in the case of zero magnetic field), which is due to the length of the different paths travelled by the free electron:

\[ \psi = \psi_1 + \psi_2 \]

where \( \psi_1 \) is the wavefunction corresponding to the first path and \( \psi_2 \) is the wavefunction from the second path. The electron in the screen is a superposition of the two paths (by quantum mechanical sense - i.e. we do not know which path the electron chose but we know it’s wavefunction as the superposition of the two paths!). Each path is just a “free electron plane wave” \( \psi_i = \langle x|p|\psi_i \rangle = Ne^{i(px/\hbar)} \). This results in the total wavefunction:

\[ \psi = \psi_1 + \psi_2 = Ne^{i(px_1/\hbar)} + Ne^{i(px_2/\hbar)} \]
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On the screen the **modulus squared** of the wavefunction is the usual Young’s double slit pattern:

\[ |\psi|^2 = |N|^2 \left( 2 + 2 \cos \frac{p(L_1 - L_2)}{\hbar} \right) = 4|N|^2 \cos^2 \frac{p(L_1 - L_2)}{2\hbar} \]

where \( L_1 \) is the length of the first path and \( L_2 \) is the length of the second path. Hence we have bright fringes for \( p\Delta L/2\hbar = m\pi \) and dark fringes for \( p\Delta L/2\hbar = (2k + 1)\pi/2 \).

Now consider switching on the magnetic field. As we know by now the electrons acquire a phase shift when a constant magnetic field is nearby (even though the particle does not experience the actual magnetic field). Switching on the magnetic field results in the phase shift of the wavefunction of electrons travelling through a path \( \gamma_1 \):

\[ \phi_1 = \mathcal{A}(\vec{r}) = \frac{2}{\hbar} \int_{\gamma_1} \vec{A} \cdot d\vec{r} \]
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\[ \varphi = m(\vec{r}) - \frac{2}{\hbar} \int_{\gamma_1} \vec{A} \cdot d\vec{r} \]
**Aharonov-Bohm effect**

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For a different path $\gamma_2$ we just replace $\gamma_1 \rightarrow \gamma_2$. The phase shift between two paths one through the first slit and the other through the second is:

$$\Delta \varphi = \varphi_1 - \varphi_2 = \frac{q}{\hbar} \int_{\gamma_1} \vec{A}(\vec{r}').d\vec{r}' - \frac{q}{\hbar} \int_{\gamma_2} \vec{A}(\vec{r}').d\vec{r}'$$

Now turning these two integrals into a single contour integral yields:

$$\Delta \varphi = \frac{q}{\hbar} \oint_{C} \vec{A}(\vec{r}').d\vec{r}'$$

where the closed loop integral runs over the corresponding two paths. By stoke’s theorem we can turn this into a surface integral to give the result $\Phi_m = B.A$. Then we see that the loop integral is zero if the two paths forming a loop do not enclose the solenoid and is non-zero when they enclose it.
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In the experimental setup above we see that the two paths do enclose the solenoid which results in the two paths acquiring different phases, and the overall phase shift due to switching on the magnetic field is:

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Hence the interference pattern on the screen gets shifted such that:

$$p\Delta L/2\hbar + \frac{q}{\hbar} B A = m\pi$$

correspond to bright fringes and

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Charged particle in an electromagnetic field

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This happens even though the electrons do not actually experience the magnetic field. This effect is due to the non-local nature of QM (i.e. electrons are not actually localised). The above analysis clearly indicates that the vector potential $\mathbf{A}$ is much more fundamental than the actual magnetic field in quantum mechanics.
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